Ng, KOK-HAUR, Corresponding author
Institute of Mathematical Sciences, Faculty of Science
University of Malaya Kuala Lumpur, Malaysia
E-mail: kokhaur@um.edu.my
Peiris SHELTON
School of Mathematics and Statistics, University of Sydney
Australia
E-mail: shelton.peiris@sydney.edu.au
Aerambamoorthy THAVANESWARAN
Department of Statistics, University of Manitoba
Winnipeg, Manitoba, Canada
E-mail: Aerambamoorthy.Thavaneswaran@umanitoba.ca
Ng, KOOI-HUAT
Department of Mathematical and Actuarial Sciences
Universiti Tunku Abdul Rahman, Malaysia
E-mail: khng@utar.edu.my

MODELLING THE RISK OR PRICE DURATIONS IN FINANCIAL MARKETS: QUADRATIC ESTIMATING FUNCTIONS AND APPLICATIONS

Abstract*. In order to minimize the associated risk in applications of duration data in financial markets, this paper considers an estimation procedure based on the theory of quadratic estimating functions (QEF). We study the associated inference problem for autoregressive conditional duration (ACD) models with nonlinear specifications to justify this approach. A Monte Carlo simulation study is carried out to asses the performance of the QEF and show that the QEF estimators outperform the linear estimating functions (LEF) estimators in almost all cases.*

Key Words: *Duration model, Quadratic Estimating Functions, Parameters, Estimation, Information matrix.*

JEL classification: **C1, C3, C4, C5**

1. Introduction

Durations between consecutive transactions play an important role in financial economics and decision making. Engle and Russell (1998) proposed a new class of duration model called autoregressive conditional duration (ACD) for modelling irregularly spaced duration data. However, many studies show that linear specifications and monotonic functions are too restrictive and hence several modifications for the ACD class have been proposed. For example: Bauwens and Giot (2000) - Logarithmic ACD (Log-ACD) models; Dufour and Engle (2000) - Box-Cox ACD (BCACD) and Exponential ACD (EXPACD) models; Zhang et al. (2001) - Threshold ACD (TACD) models; Bauwens and Giot (2003) - asymmetric ACD model; Bauwens and Veredas (2004) - stochastic conditional duration (SCD) model; Fernandes and Grammig (2006) - augmented autoregressive conditional duration (AACD) model; and Meitz and Terasvirta (2006) - smooth-transition threshold (ST-) ACD model. Several good reviews on various ACD models with applications can be found in Pacurar (2008) and Hautsch (2012).

The maximum likelihood and quasi maximum likelihood are widely used in parameter estimation of ACD models. These methods do not work well unless the distribution of errors completely or approximately known. Therefore, a semiparametric approach based on the theory of Estimating Function (EF) due to Godambe (1985) has been successfully applied in many financial economic time series models including ACD models. For example, Thavaneswaran and Peiris (1996), Chandra and Taniguchi (2001) and Merkouris (2007) used the linear estimating function (LEF) approach for estimation of nonlinear time series. Allen et al. (2013a, b) and Ng and Peiris (2013) have reported the use of LEF in estimation of ACD models. David and Turtle (2000) applied the combined EF approach in Autoregressive Conditional Heteroscedasticity (ARCH) models.

The main objective of this paper is to develop a new estimation procedure for ACD models with nonlinear specifications based on the theory of quadratic estimating functions (QEF) due to Liang et al. (2011). Thavaneswaran et al. (2012) applied the QEF in Random Coefficient Autoregressive (RCA) models with Generalized Autoregressive Heteroscedasticity (GARCH) innovations and derived a number of interesting and elegant results. The QEF is essentially the same as combined estimating function proposed by David and Turtle (2000). However, the QEF method can be applied to any time series model.

In this paper, we apply the theory of QEF in estimation of Log-ACD and BCACD models and derive their corresponding information matrices to aid statistical inference. The motivation for this work hails from the fact that Log-ACD and BCACD models are potentially more flexible and less restriction on parameters than in LINACD (linear ACD) models. See for example, Bauwens and Giot (2000, 2003), Bauwens et al. (2004), Bauwens et al. (2008) and Allen et al. (2008, 2009) for related discussions.

The remainder of this paper is organized as follows. Section 2 reports basic results of the class of LEF and reviews the theory of optimal QEF and the Section 3 provides the optimal QEF and the corresponding information matrices for Log-ACD models and BCACD models. The Section 4 gives simulation results to verify the corresponding theoretical results. Finally, we give the some concluding remarks in Section 5.

2. Basic Results

2.1 Notation

Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ is a discrete valued stochastic process and we are interested in fitting a suitable model for this sample of size n. Let Θ be the class of probability distributions $\mathbf{F}$ on $\Re^n$ and $\boldsymbol{\theta} = \boldsymbol{\theta}(F)$, $\mathbf{F} \in \Theta$ be a vector of real parameters.

Let $E_{i-1,\mathbf{F}}(\cdot)$ be the conditional expectation holding the first $i-1$ values $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{i-1}$ fixed. For simplicity, let $E_{i-1,\mathbf{F}}(\cdot) \equiv E_{i-1}(\cdot)$ and let $E_{\mathbf{F}}(\cdot) \equiv E(\cdot)$.

Let $\mathbf{h}_i(\cdot)$ be a real valued function of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_i$ and $\boldsymbol{\theta}$ such that

$$E_{i-1,\mathbf{F}}[\mathbf{h}_i(\cdot)] = \mathbf{0}, \; (\; i = 1,2,\cdots,n, \; \mathbf{F} \in \Theta \;) \tag{1}$$

and

$$E(\mathbf{h}_i(\cdot)\mathbf{h}_j^{'}(\cdot)) = \mathbf{0}, \; (i \neq j). \tag{2}$$

Suppose that $\mathbf{g}(\mathbf{x};\boldsymbol{\theta})$ is a real valued function of the random variate $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ and the parameter $\boldsymbol{\theta}$, satisfying some regularity conditions (see e.g. Godambe (1985)). Then the function $\mathbf{g}(\mathbf{x};\boldsymbol{\theta})$ with $E[\mathbf{g}(\mathbf{x};\boldsymbol{\theta})] = \mathbf{0}$ is called a regular unbiased estimating function. Among all regular unbiased estimating functions $\mathbf{g}(\mathbf{x};\boldsymbol{\theta})$, $\mathbf{g}^*(\mathbf{x};\boldsymbol{\theta})$ is said to be optimum if

$$E[\mathbf{g}(\mathbf{x};\boldsymbol{\theta})\mathbf{g}(\mathbf{x};\boldsymbol{\theta})^{'}] \Big/ \left\{ E\left(\left[\frac{\partial \mathbf{g}(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]\right) E\left(\left[\frac{\partial \mathbf{g}(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]\right)^{'} \right\} \tag{3}$$

is minimized for all $\mathbf{F} \in \Theta$ at $\mathbf{g}(\mathbf{x};\boldsymbol{\theta}) = \mathbf{g}^*(\mathbf{x};\boldsymbol{\theta})$. An optimal estimate of $\boldsymbol{\theta}$ is obtained by solving the optimum estimating equation $\mathbf{g}^*(\mathbf{x};\boldsymbol{\theta}) = \mathbf{0}$.

Consider the class of linear unbiased estimating functions G formed by

$$\mathbf{g}(\mathbf{x};\boldsymbol{\theta}) = \sum_{i=1}^{n} \mathbf{a}_{i-1}\mathbf{h}_i, \tag{4}$$

where $\mathbf{a}_{i-1}$ is a suitably chosen function of the random variates $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{i-1}$ and the parameter $\boldsymbol{\theta}$ for all $i = 1,2,\cdots,n$. From Eq.(4) it is clear that, $E[\mathbf{g}(\mathbf{x};\boldsymbol{\theta})] = \mathbf{0}$, for all $\mathbf{g}(\mathbf{x};\boldsymbol{\theta}) \in G$.

Following the optimal theorem of Godambe (1985), the function $\mathbf{g}^*(\mathbf{x};\boldsymbol{\theta})$ minimizing Eq.(3) is given by

$$\mathbf{g}^*(\mathbf{x};\boldsymbol{\theta}) = \sum_{i=1}^{n} \mathbf{a}_{i-1}^{*}\mathbf{h}_i \;,$$

where $\mathbf{a}_{i-1}^{*} = \left(E_{i-1}\left[\frac{\partial \mathbf{h}_i}{\partial \boldsymbol{\theta}} \right] \right)' \Big/ E_{i-1}[\mathbf{h}_i \mathbf{h}_i']$.

An optimal estimate of $\boldsymbol{\theta}$ (in the sense of Godambe (1985)) can be obtained by solving the equation(s) $\mathbf{g}^{*}(\mathbf{x};\boldsymbol{\theta}) = \mathbf{0}$.

2.2 Quadratic Estimating Functions - QEF

Suppose that the following conditional moments exist for the stochastic process $\{x_i, i = 1,2,\cdots,n\}$:

$\mu_i(\boldsymbol{\theta}) = E[x_i \mid F_{i-1}]$, $\sigma_i^2(\boldsymbol{\theta}) = E[(x_i - \mu_i(\boldsymbol{\theta}))^2 \mid F_{i-1}] = \text{Var}[x_i \mid F_{i-1}]$,

$\gamma_i(\boldsymbol{\theta}) = \dfrac{1}{\sigma_i^3(\boldsymbol{\theta})} E[(x_i - \mu_i(\boldsymbol{\theta}))^3 \mid F_{i-1}]$, and

$\kappa_i(\boldsymbol{\theta}) = \dfrac{1}{\sigma_i^4(\boldsymbol{\theta})} E[(x_i - \mu_i(\boldsymbol{\theta}))^4 \mid F_{i-1}] - 3$.

where F_{i-1} is the information set $\{x_{i-1}, x_{i-2}, \cdots, x_1\}$. We further assume that the skewness $\gamma_i(\boldsymbol{\theta})$ and the excess kurtosis $\kappa_i(\boldsymbol{\theta})$ of the standardized variable x_i do not contain any additional parameter. In order to estimate the parameter $\boldsymbol{\theta}$ based on the observations $x_1, x_2, \cdots, x_n$, we consider the following two classes of martingale differences:

$\{m_i(\boldsymbol{\theta}) = x_i - \mu_i(\boldsymbol{\theta}), i = 1,2,\cdots,n\}$, and $\{s_i(\boldsymbol{\theta}) = m_i^2(\boldsymbol{\theta}) - \sigma_i^2(\boldsymbol{\theta}), i = 1,2,\cdots,n\}$.

Define the following notation for convenience and for later reference:

- $\langle m \rangle_i = E[m_i^2(\boldsymbol{\theta}) \mid F_{i-1}] = \text{Var}[m_i(\boldsymbol{\theta}) \mid F_{i-1}] = \sigma_i^2(\boldsymbol{\theta})$.
- $\langle s \rangle_i = E[s_i^2(\boldsymbol{\theta}) \mid F_{i-1}] = \sigma_i^4(\boldsymbol{\theta})(\kappa_i(\boldsymbol{\theta}) + 2)$.
- $\langle m, s \rangle_i = E[m_i(\boldsymbol{\theta}) s_i(\boldsymbol{\theta}) \mid F_{i-1}] = \sigma_i^3(\boldsymbol{\theta})\gamma_i(\boldsymbol{\theta})$.

Now we state the following for the optimal estimating functions based on the above martingale differences.

Lemma 1: The optimal EF based on the above martingale differences

(i) $m_i(\boldsymbol{\theta})$ and $s_i(\boldsymbol{\theta})$ are $g_M^{*}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} \dfrac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \dfrac{m_i(\boldsymbol{\theta})}{\langle m \rangle_i}$ and

$g_S^{*}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} \dfrac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \dfrac{s_i(\boldsymbol{\theta})}{\langle s \rangle_i}$,

(ii) The corresponding information associated with $g_M^*(\boldsymbol{\theta})$ and $g_S^*(\boldsymbol{\theta})$ are $\mathbf{I}_{g_M^*}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \frac{1}{\langle m \rangle_i}$ and $\mathbf{I}_{g_S^*}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \frac{1}{\langle s \rangle_i}$, respectively.

Now we state the following theorem due to Liang et al. (2011):

Theorem 1: Suppose that $\{x_i\}$ is a stochastic process with finite 4th order moments. The corresponding QEF has the form

$$\mathbf{g} = \left\{ \mathbf{g}_Q(\mathbf{x};\boldsymbol{\theta}) : \mathbf{g}_Q(\boldsymbol{\theta}) = \sum_{i=1}^{n} (\mathbf{a}_{i-1} m_i(\boldsymbol{\theta}) + \mathbf{b}_{i-1} s_i(\boldsymbol{\theta})) \right\}$$

(i) the optimal estimating function is given by

$\mathbf{g}_Q^*(\mathbf{x};\boldsymbol{\theta}) = \sum_{i=1}^{n} (\mathbf{a}_{i-1}^* m_i(\boldsymbol{\theta}) + \mathbf{b}_{i-1}^* s_i(\boldsymbol{\theta}))$, where

$$\mathbf{a}_{i-1}^* = \left(1 - \frac{\langle m, s \rangle_i^2}{\langle m \rangle_i \langle s \rangle_i}\right)^{-1} \left(-\left(\frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right) \frac{1}{\langle m \rangle_i} + \left(\frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right) \frac{\langle m, s \rangle_i}{\langle m \rangle_i \langle s \rangle_i} \right),$$

and

$$\mathbf{b}_{i-1}^* = \left(1 - \frac{\langle m, s \rangle_i^2}{\langle m \rangle_i \langle s \rangle_i}\right)^{-1} \left(\left(\frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right) \frac{\langle m, s \rangle_i}{\langle m \rangle_i \langle s \rangle_i} - \left(\frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right) \frac{1}{\langle s \rangle_i} \right).$$

An optimal estimate of $\boldsymbol{\theta}$ can be obtained by solving the equation(s) $\mathbf{g}_Q^*(\mathbf{x};\boldsymbol{\theta}) = \mathbf{0}$.

(ii) the information $\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta})$ is given by $\mathrm{I}_{\mathbf{g}_Q^*}(\mathbf{x};\boldsymbol{\theta}) = \sum_{i=1}^{n} \left(1 - \frac{\langle m, s \rangle_i^2}{\langle m \rangle_i \langle s \rangle_i}\right)^{-1} (A_i + B_i)$

where $A_i = \frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \frac{1}{\langle m \rangle_i} + \frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \frac{1}{\langle s \rangle_i}$,

$$B_i = -\left(\frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \right) \left(\frac{\partial \sigma_i^2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \mu_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \right) \frac{\langle m, s \rangle_i}{\langle m \rangle_i \langle s \rangle_i}$$

Proof is given in Liang (2011), pp. 4.

The Section 3 considers two main applications of the previous results for Log-ACD and BCACD models.

3. Applications of the QEF

We first consider the class of Log-ACD or known as Log-ACD (p,q) models.

3.1 Log-ACD (p,q) Model

Define ψ_i as the logarithm of the conditional expectation of x_i, so that:

$$\psi_i = \ln E[x_i \mid x_{i-1}, x_{i-2}, \cdots, x_1] = \ln E[x_i \mid F_{i-1}], \tag{5}$$

where F_{i-1} is the information set available at the $(i-1)$th trade. Then, the Log-ACD (p,q) model is defined by:

$$x_i = e^{\psi_i} \varepsilon_i, \tag{6}$$

$$\psi_i = \omega + \sum_{j=1}^{p} \alpha_j \ln(x_{i-j}) + \sum_{k=1}^{q} \beta_k \psi_{i-k} . \tag{7}$$

where ε_i is a sequence of independently and identically distributed (iid) non-negative random variable's with mean μ_ε, variance σ_ε^2, skewness γ_ε and excess kurtosis κ_ε and ε_i is independent of F_{i-1}. There are no positivity restrictions required on the parameters ω, α_j and β_k in ψ_i, since $e^{\psi_i} > 0$ guarantees $x_i > 0$. Analytical expressions for some moments of Log-ACD (p,q) models can be found in Bauwens et al. (2008).

To find the QEF estimates for Log-ACD (p,q) models, let $\mu_i(\boldsymbol{\theta}) = \mu_\varepsilon e^{\psi_i}$, $\sigma_i^2(\boldsymbol{\theta}) = \sigma_\varepsilon^2 e^{2\psi_i}$, $\gamma_i(\boldsymbol{\theta}) = \gamma_\varepsilon$ and $\kappa_i(\boldsymbol{\theta}) = \kappa_\varepsilon$. It is clear that the corresponding $\langle m \rangle_i = \sigma_\varepsilon^2 e^{2\psi_i}$, $\langle s \rangle_i = (\kappa_\varepsilon + 2)\sigma_\varepsilon^4 e^{4\psi_i}$ and $\langle m, s \rangle_i = \gamma_\varepsilon \sigma_\varepsilon^3 e^{3\psi_i}$. In order to estimate the parameters in $\boldsymbol{\theta} = (\omega, \alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_q)$, we use the approach of the LEF and QEF. The optimal LEF based on $m_i(\boldsymbol{\theta})$ and $s_i(\boldsymbol{\theta})$ respectively are:

$$\mathbf{g}_M^*(\mathbf{x};\boldsymbol{\theta}) = -\sum_{i=1}^{n} \frac{\mu_\varepsilon}{\sigma_\varepsilon^2} \left(\frac{1}{e^{\psi_i}} \right) \frac{\partial \psi_i}{\partial \boldsymbol{\theta}} m_i(\boldsymbol{\theta}) \text{ and}$$

$$\mathbf{g}_S^*(\mathbf{x};\boldsymbol{\theta}) = -\sum_{i=1}^{n} \frac{2}{(\kappa_\varepsilon + 2)\sigma_\varepsilon^2} \left(\frac{1}{e^{2\psi_i}} \right) \frac{\partial \psi_i}{\partial \boldsymbol{\theta}} s_i(\boldsymbol{\theta}) .$$

The optimal QEF is given by:

$$\mathbf{g}_Q^*(\mathbf{x};\boldsymbol{\theta}) = \left(1 - \frac{\gamma_\varepsilon^2}{(\kappa_\varepsilon + 2)}\right)^{-1}$$

$$\sum_{i=1}^{n}\left(\frac{(-\mu_\varepsilon(\kappa_\varepsilon + 2) + 2\gamma_\varepsilon\sigma_\varepsilon)}{\sigma_\varepsilon^2(\kappa_\varepsilon + 2)e^{\psi_i}}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}}m_i(\boldsymbol{\theta}) + \frac{(\mu_\varepsilon\gamma_\varepsilon - 2\sigma_\varepsilon)}{\sigma_\varepsilon^3(\kappa_\varepsilon + 2)e^{2\psi_i}}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}}s_i(\boldsymbol{\theta})\right)$$

and

$$\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta}) = \left(1 - \frac{\gamma_\varepsilon^2}{(\kappa_\varepsilon + 2)}\right)^{-1}\left(\mathrm{I}_{\mathbf{g}_M^*}(\boldsymbol{\theta}) + \mathrm{I}_{\mathbf{g}_S^*}(\boldsymbol{\theta}) - \frac{4\gamma_\varepsilon\mu_\varepsilon}{\sigma_\varepsilon(\kappa_\varepsilon + 2)}\sum_{i=1}^{n}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}'}\right),$$

where

$$\mathrm{I}_{\mathbf{g}_M^*}(\boldsymbol{\theta}) = \frac{\mu_\varepsilon^2}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}'}, \quad \mathrm{I}_{\mathbf{g}_S^*}(\boldsymbol{\theta}) = \frac{4}{(\kappa_\varepsilon + 2)}\sum_{i=1}^{n}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}'},$$

$$\frac{\partial\psi_i}{\partial\omega} = \left(1 + \sum_{k=1}^{q}\beta_k\frac{\partial\psi_{i-k}}{\partial\omega}\right), \quad \frac{\partial\psi_i}{\partial\alpha_l} = \left(\ln x_{i-l} + \sum_{k=1}^{q}\beta_k\frac{\partial\psi_{i-k}}{\partial\alpha_l}\right) \text{ for } l = 1,2,\cdots,p \quad \text{and}$$

$$\frac{\partial\psi_i}{\partial\beta_m} = \left(\psi_{i-m} + \sum_{k=1}^{q}\beta_k\frac{\partial\psi_{i-k}}{\partial\beta_m}\right) \text{ for } m = 1,2,\cdots,q.$$

It follows from Lindsay (1985) that the asymptotic variances of the resulting estimators are the inverse of the information matrices $\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta})$, $\mathrm{I}_{\mathbf{g}_M^*}(\boldsymbol{\theta})$ and $\mathrm{I}_{\mathbf{g}_S^*}(\boldsymbol{\theta})$. Hence, the estimator obtained from a more informative estimating equation is asymptotically more efficient.

The information gain in using QEF ($\mathbf{g}_Q^*(\mathbf{x};\boldsymbol{\theta})$) over LEF ($\mathbf{g}_M^*(\mathbf{x};\boldsymbol{\theta})$) is

$$\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta}) - \mathrm{I}_{\mathbf{g}_M^*}(\boldsymbol{\theta}) = \frac{(2\sigma_\varepsilon - \mu_\varepsilon\gamma_\varepsilon)^2}{\sigma_\varepsilon^2(\kappa_\varepsilon + 2 - \gamma_\varepsilon^2)}\sum_{i=1}^{n}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}'},$$

and information gain in using QEF($\mathbf{g}_Q^*(\mathbf{x};\boldsymbol{\theta})$) over LEF($\mathbf{g}_S^*(\mathbf{x};\boldsymbol{\theta})$) is

$$\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta}) - \mathrm{I}_{\mathbf{g}_S^*}(\boldsymbol{\theta}) = \frac{(\mu_\varepsilon(\kappa_\varepsilon + 2) - 2\sigma_\varepsilon\gamma_\varepsilon)^2}{\sigma_\varepsilon^2(\kappa_\varepsilon + 2 - \gamma_\varepsilon^2)(\kappa_\varepsilon + 2)}\sum_{i=1}^{n}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}}\frac{\partial\psi_i}{\partial\boldsymbol{\theta}'}.$$

Now we consider the BCACD or known as BCACD(p,q) models for illustration.

3.2 BCACD (p,q) model

Let

$$\psi_i = E[x_i \mid x_{i-1}, x_{i-2}, \cdots, x_1] = E[x_i \mid F_{i-1}]. \tag{8}$$

Then, the BCACD(p,q) model for the variable x_i is defined as

$$x_i = \psi_i \varepsilon_i, \tag{9}$$

$$\ln \psi_i = \omega + \sum_{j=1}^{p} \alpha_j \varepsilon_{i-j}^{\delta} + \sum_{k=1}^{q} \beta_k \ln \psi_{i-k}, \tag{10}$$

where ω, δ, α_j and β_k are parameters and ε_i as defined before. In the paper by Dufour and Engle (2000), they have pointed out the following two main drawbacks of LINACD model (i) a constraint on the parameters to ensure non-negative durations, and (ii) the assumption of linearity is not appropriate in many applications.

In order to estimate the parameter vector $\boldsymbol{\theta} = (\omega, \alpha_1, \alpha_2, \cdots, \alpha_p, \delta, \beta_1, \beta_2, \cdots, \beta_q)$ of BCACD(p,q) models, we use the methods in Sections 2 and 3. We found that the results for $\mathbf{g}_M^*(\mathbf{x};\boldsymbol{\theta})$, $\mathbf{g}_S^*(\mathbf{x};\boldsymbol{\theta})$, $\mathbf{g}_Q^*(\mathbf{x};\boldsymbol{\theta})$, $\mathrm{I}_{\mathbf{g}_M^*}(\boldsymbol{\theta})$, $\mathrm{I}_{\mathbf{g}_S^*}(\boldsymbol{\theta})$ and $\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta})$ are basically the same as LINACD(p,q) models given in Liang et al. (2011) except the term ψ_i^2 is not include in each equation. The partial derivatives of ψ_i with respect to each parameter of BCACD(p,q) models are given as

$$\frac{\partial \psi_i}{\partial \omega} = \psi_i \left[1 + \sum_{k=1}^{q} \frac{\beta_k}{\psi_{i-k}} \frac{\partial \psi_{i-k}}{\partial \omega} \right],$$

$$\frac{\partial \psi_i}{\partial \delta} = \psi_i \left[\sum_{j=1}^{p} \alpha_j \varepsilon_{i-j}^{\delta} \ln \varepsilon_{i-j} + \sum_{k=1}^{q} \frac{\beta_k}{\psi_{i-k}} \frac{\partial \psi_{i-k}}{\partial \delta} \right],$$

$$\frac{\partial \psi_i}{\partial \alpha_l} = \psi_i \left[\varepsilon_{i-l}^{\delta} + \sum_{k=1}^{q} \frac{\beta_k}{\psi_{i-k}} \frac{\partial \psi_{i-k}}{\partial \alpha_l} \right] \text{ for } l = 1,2,\cdots,p,$$

and $$\frac{\partial \psi_i}{\partial \beta_m} = \psi_i \left[\ln \psi_{i-m} + \sum_{k=1}^{q} \frac{\beta_k}{\psi_{i-k}} \frac{\partial \psi_{i-k}}{\partial \beta_m} \right] \text{ for } m = 1,2,\cdots,q.$$

When ε_i follows a standardized weibull distribution with parameter $\alpha = 1$, this distribution reduces to an exponential distribution and gives $\mu_\varepsilon = 1$, $\sigma_\varepsilon^2 = 2$, $\gamma_\varepsilon = 2$ and $\kappa_\varepsilon = 6$, we can show that $\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta}) = \mathrm{I}_{\mathbf{g}_M^*}(\boldsymbol{\theta}) > \mathrm{I}_{\mathbf{g}_S^*}(\boldsymbol{\theta})$. In general, when ε_i follows a standardized distribution, that is a distribution with unit expectation, then $\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta}) \geq \mathrm{I}_{\mathbf{g}_M^*}(\boldsymbol{\theta})$ and $\mathrm{I}_{\mathbf{g}_Q^*}(\boldsymbol{\theta}) \geq \mathrm{I}_{\mathbf{g}_S^*}(\boldsymbol{\theta})$.

3. Monte Carlo Simulation

A Monte Carlo simulation is carried out to verify the theoretical results given in Section 3. We compare the finite sample performance for QEF and LEF methods using the Log-ACD (1,1) with various popular error distributions, namely the standardized Exponential distribution, standardized Weibull distribution, standardized Generalized Gamma distribution and standardized Lognormal distribution.

- Simulate a time series of length $n = 500$ using Log-ACD (1,1) with $\omega = 0.20$, $\alpha_1 = 0.30$, $\beta_1 = 0.40$, $\psi_1 = 0.50$ and error distribution is standardized Exponential distribution.
- Estimate the parameters of the model using QEF and LEF methods.
- The procedure is repeated for $N = 2000$ replications.
- Finally, we compute the mean, bias, standard error (SE) and root mean squared error (RMSE) of the parameter estimates.
- The whole procedure will be repeated for different error distributions.

Table 1 shows the results for sample sizes $n = 500$ and $n = 2000$ with various error distributions. For Log-ACD (1,1) models with sample size $n = 500$, the QEF method gives smaller bias for all estimates than the LEF method. As expected, the RMSEs for all estimates when the error distribution is exponential are comparable for the QEF and LEF methods. When the error distribution is not exponential distribution, the QEF method gives smaller estimated standard errors than the LEF method. As the sample size increases to $n = 2000$, it is clear that the estimated standard errors have been reduced. It can be seen that the values of the estimated bias of the estimates are close to their true values in both the QEF and LEF methods.

Table 1: Estimation results for the Log-ACD (1,1) models with various distribution obtained from sample size $n = 500$ with $N = 2000$ simulation runs ($\omega = 0.20$, $\alpha_1 = 0.30$, $\beta_1 = 0.40$ and $\psi_1 = 0.50$). Data are generated from various distributions as given in column 1. Values in parentheses are obtained from sample size $n = 2000$.

True Distribution	$\hat{\omega}$		$\hat{\alpha}_1$		$\hat{\beta}_1$	
Exponential	**QEF**	**EF**	**QEF**	**EF**	**QEF**	**EF**
Mean	0.1994 (0.2000)	0.1989 (0.2002)	0.3001 (0.3002)	0.2983 (0.3003)	0.3905 (0.3997)	0.3892 (0.3967)
Bias	-0.0006 (0.0000)	-0.0011 (0.0002)	0.0001 (0.0002)	-0.0017 (0.0003)	-0.0095 (-0.0003)	-0.0108 (-0.0033)
SE	0.0223 (0.0106)	0.0216 (0.0109)	0.0343 (0.0164)	0.0334 (0.0168)	0.0790 (0.0387)	0.0780 (0.0396)
RMSE	0.0223 (0.0106)	0.0216 (0.0109)	0.0343 (0.0164)	0.0334 (0.0168)	0.0795 (0.0388)	0.0788 (0.0397)
Weibull with $\gamma = 2.0$	**QEF**	**EF**	**QEF**	**EF**	**QEF**	**EF**
Mean	0.2045 (0.2005)	0.2059 (0.2009)	0.3002 (0.2999)	0.2992 (0.2996)	0.3892 (0.3988)	0.3880 (0.3982)
Bias	0.0045 (0.0005)	0.0059 (0.0009)	0.0002 (-0.0001)	-0.0008 (-0.0004)	-0.0108 (-0.0012)	-0.0120 (-0.0018)
SE	0.0339 (0.0159)	0.0355 (0.0171)	0.0344 (0.0168)	0.0359 (0.0180)	0.0810 (0.0385)	0.0846 (0.0412)
RMSE	0.0341 (0.0159)	0.0360 (0.0171)	0.0344 (0.0168)	0.0359 (0.0180)	0.0817 (0.0385)	0.0855 (0.0412)
Weibull with $\gamma = 3.0$	**QEF**	**EF**	**QEF**	**EF**	**QEF**	**EF**
Mean	0.2057 (0.2016)	0.2088 (0.2016)	0.2992 (0.3000)	0.2983 (0.2994)	0.3901 (0.3970)	0.3862 (0.3977)
Bias	0.0057 (0.0016)	0.0088 (0.0016)	-0.0008 (0.0000)	-0.0017 (-0.0006)	0.0099 (-0.0030)	-0.0138 (-0.0023)
SE	0.0371 (0.0179)	0.0413 (0.0202)	0.0337 (0.0166)	0.0361 (0.0186)	0.0781 (0.0378)	0.0851 (0.0430)
RMSE	0.0375 (0.0180)	0.0422 (0.0202)	0.0337 (0.0166)	0.0361 (0.0186)	0.0787 (0.0379)	0.0862 (0.0431)
G.Gamma with $\alpha = 4.0$**,** $\kappa = 3.0$	**QEF**	**EF**	**QEF**	**EF**	**QEF**	**EF**
Mean	0.2112 (0.2078)	0.2122 (02075)	0.2990 (0.3022)	0.2991 (0.3022)	0.3835 (0.3858)	0.3821 (0.3856)
Bias	0.0112 (0.0078)	0.0122 (0.0075)	-0.0010 (0.0022)	-0.0009 (0.0022)	-0.0165 (-0.0142)	-0.0179 (-0.0144)
SE	0.0487 (0.0237)	0.0507 (0.0247)	0.0398 (0.0203)	0.0419 (0.0214)	0.0931 (0.0460)	0.0973 (0.0484)
RMSE	0.0500 (0.0249)	0.0521 (0.0258)	0.0398 (0.0204)	0.0419 (0.0215)	0.0945 (0.0482)	0.0989 (0.0503)

Lognormal With $\sigma^2 = 0.4$	**QEF**	**EF**	**QEF**	**EF**	**QEF**	**EF**
Mean	0.2071 (0.2019)	0.2082 (0.2016)	0.2985 (0.2996)	0.2980 (0.2997)	0.3854 (0.3960)	0.3824 (0.3963)
Bias	0.0071 (0.0019)	0.0082 (0.0016)	-0.0015 (-0.0004)	-0.0020 (-0.0003)	-0.0146 (-0.0040)	-0.0176 (-0.0037)
SE	0.0434 (0.0207)	0.0468 (0.0220)	0.0441 (0.0226)	0.0479 (0.0237)	0.1066 (0.0523)	0.1158 (0.0551)
RMSE	0.0440 (0.0208)	0.0475 (0.0220)	0.0441 (0.0226)	0.0480 (0.0237)	0.1076 (0.0524)	0.1171 (0.0552)

Figures 1 to 3 show the histograms of parameter estimates $\hat{\boldsymbol{\theta}} = (\hat{\omega}, \hat{\alpha}_1, \hat{\beta}_1)$ obtained by QEF method for Log-ACD (1,1) model when the true distribution follows standardized weibull distribution (with $\gamma = 3.0$) with sample size of $n = 500$. These histograms show that $\hat{\boldsymbol{\theta}} = (\hat{\omega}, \hat{\alpha}_1, \hat{\beta}_1)$ follow approximate a normal distribution with mean $\boldsymbol{\theta}$ and variance $(I_{g_Q^*}(\boldsymbol{\theta}))^{-1}$.

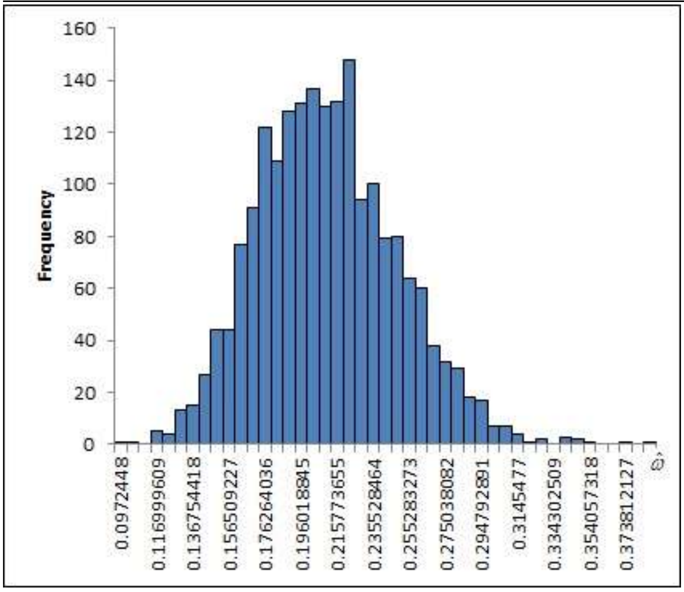

Figure 1. The histogram for $\hat{\omega}$ obtained by the QEF method when the true distribution follows standardized weibull distribution with $\gamma = 3.0$ ($\omega = 0.2, n = 500, N = 2000$).

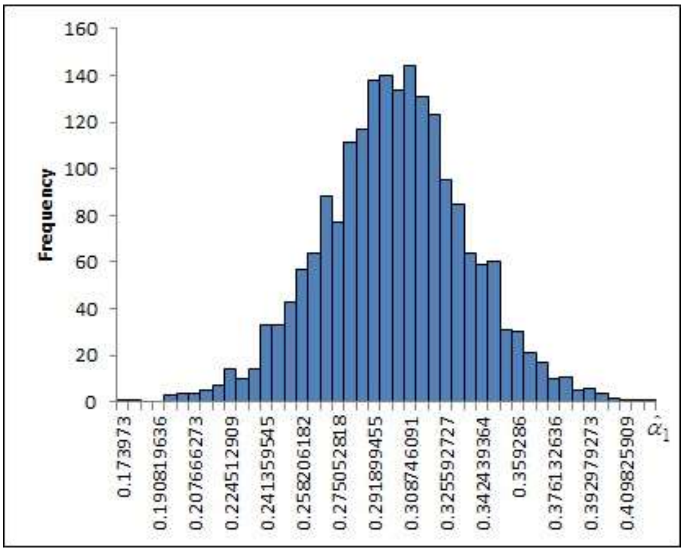

Figure 2. The histogram for $\hat{\alpha}_1$ obtained by the QEF method when the true distribution follows standardized weibull distribution with $\gamma = 3.0$ ($\alpha_1 = 0.3, n = 500, N = 2000$).

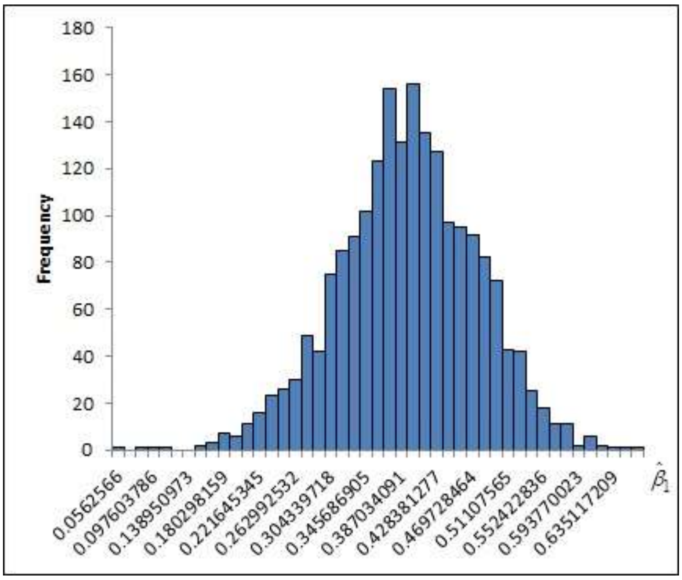

Figure 3. The histogram for $\hat{\beta}_1$ obtained by the QEF method when the true distribution follows standardized weibull distribution with $\gamma = 3.0$ ($\hat{\beta}_1 = 0.4, n = 500, N = 2000$).

5. Conclusion

In this paper, we have used semiparamteric approaches based on QEF and LEF for the classes of Log-ACD and BCACD models. The properties of these proposed estimators have been investigated. Theoretical results of the QEF estimates have been used to develop inferential results of the corresponding estimators. Based on a large simulation study, we have shown that the QEF have smaller standard errors than those of the LEF and the distribution of the estimates is approximately normal. Thus, the QEF estimates are more reliable than those of the LEF and, hence, useful in modelling and forecasting of duration data at minimum risk.

Acknowledgement

The first author greatly acknowledges that this work is partially supported by the UMRG grant nos: RG174-11AFR and RG260-13AFR of the University of Malaya. The second author thanks the University of Malaya and Institute of Mathematical Sciences for hospitality during his visit. The final of the paper was completed while the first author was at University of Sydney in February 2014.

REFERENCES

[1] **Allen, D., Chan, F., McAleer, M., Peiris, M.S. (2008),** ***Finite Sample Properties of QMLE for the Log-ACD Model: Application to Australian Stocks***. *Journal of Econometrics*, **147**(**1**), 163-185;
[2] **Allen, D., Lazarov, Z., McAleer, M., Peiris, M.S. (2009),** ***Comparison of Alternative ACD Models via Density and Interval Forecasts: Evidence from the Australian Stock Market****; Mathematics and Computers in Simulation*, **79**(**8**), 2535-2555;
[3] **Allen, D., Ng, K.H., Peiris, S. (2013a),** ***Estimating and Simulating Weibull Models of Risk or Price Durations: An application to ACD Models***. *North American Journal of Economics and Finance*, **25**, 214-225;
[4] **Allen, D., Ng, K.H., Peiris, S. (2013b),** ***The Efficient Modeling of High Frequency Transaction Data: A New Application of Estimating Functions in Financial Economics***. *Economics Letters*, **120**(**1**), 177-122;
[5] **Bauwens, L., Giot, P. (2000),** ***The Logarithmic ACD Model: An Application to the Bid-ask Quote Process of Three NYSE Stocks***. *Annales D'Economie et de Statistique*, **60**, 117-145;
[6] **Bauwens, L., Giot, P. (2003),** ***Asymmetric ACD Models: Introducing Price Information in ACD Models***. *Empirical Economics*, **28**(**4**), 709-731;
[7] **Bauwens, L., Giot, P.,Grammig, J., Veredas, D. (2004),** ***A Comparison of Financial Duration Models via Density Forecasts***. *International Journal of Forecasting*, **20**(**4**), 589-609;
[8] **Bauwens, L., Veredas, D. (2004),** ***The Stochastic Conditional Duration Model: A Latent Variable Model for the Analysis of Financial Durations***. *Journal of Econometrics*, **119**(**2**), 381-412;
[9**] Bauwens, L., Galli, F., Giot, P. (2008),** ***The Moments of Log-ACD Models*****.** *Quantative and Qualitative Analysis in Social Sciences*, **2**(**1**), 1-28;
[10] **Chandra, S.A, Taniguchi, M. (2001),** ***Estimating Functions for Nonlinear Time Series Models***. *Annals of the Institute Statistic Mathematics*, **53**(**1**), 125-141;
[11] **David, X. Li., Turtle, H.J. (2000),** ***Semiparametric ARCH Models: An Estimating Function Approach***. *Journal of Business & Economic Statistics*, **18**(**2**), 174-186;
[12] **Dufour, A., Engle, R.F. (2000),** ***The ACD Model: Predictability of the Time between Consecutive Trades***. Discussion Papers in Finance, ISMA Centre, 59;
[13] **Engle, R.F., Russell, J.R. (1998*), Autoregressive Conditional Duration: A New Model for Irregularly Spaced Transaction Data***. *Econometrica*, **66**(**5**), 1127-1162;
[14] **Fernandes, M., Grammig, J. (2006),** ***A Family of Autoregressive Conditional Duration Models***. *Journal of Econometrics*, **130**(**1**), 1-23;
[15] **Godambe, V.P. (1985),** ***The Foundations of Finite Sample Estimation in Stochastic Processes***. *Biometrika*, **72**(**2**), 419-428;

[16] **Hautsch, N. (2012),** ***Econometrics of Financial High-frequency Data***. (1st ed.). *New York: Springer*;
[17] **Liang, Y., Thavaneswaran, A., Abraham, B. (2011),** ***Joint Estimation Using Quadratic Estimating Function***. *Journal of Probability and Statistics*, http://dx.doi.org/10.1155/2011/372512;
[18] **Lindsay, B.G. (1985),** ***Using Empirical Partially Bayes Inference for Increased Efficiency***. *Annals of Statistics*, **13**(**3**), 914-931;
[19] **Meitz, M., Terasvirta, T. (2006),** ***Evaluating Models of Autoregressive Conditional Duration***. *Journal of Business and Economic Statistics*, **24**(**1**), 104-124;
[20] **Merkouris, T.(2007),** ***Transform Martingale Estimating Functions***. *Annals of Statistics* **35**(**5**), 1975-2000;
[21] Ng, K.H., Peiris, S. (2013), *Modelling High Frequency Transaction Data in Financial Economics: A Comparative Study Based on Simulations*. *Economic Computation and Economics Cybernetics Studies and Research*, **47**(**2**), 189-202;
[22**] Pacurar, M. (2008),** ***Autoregressive Conditional Duration Models in Finance: A Survey of the Theoretical and Empirical Literature***. *Journal of Economic Surveys*, **22**(**4**), 711-751;
[23] **Thavaneswaran, A., Peiris, M.S. (1996),** ***Nonparametric Estimation for some Nonlinear Models***. *Statistics and Probability Letters*, **28**(**3**), 227-233;
[24] **Thavaneswaran, A., Liang, Y., Frank, J. (2012),** ***Inference for Random Coefficient Volatility Models***. *Statistics and Probability Letters*, **82**(**12**), 2086-2090;
[25] **Zhang, M.Y., Russell, J.R., Tsay, R.S. (2001),** ***A Non-linear Autoregressive Conditional Duration Model with Applications to Financial Transaction Data.*** *Journal of Econometrics*, **104**(**1**), 179-207.